Importance Analysis of Multi-State Systems based on Integrated Direct Partial Logic Derivatives

Miroslav Kvassay, Elena Zaitseva, Jozef Kostolny, Vitaly Levashenko
Faculty of Management Science and Informatics
University of Zilina
Zilina, Slovakia
{miroslav.kvassay, elena.zaitseva, jozef.kostolny, vitaly.levashenko}@fri.uniza.sk

***Abstract*—Reliability is very important characteristic of systems. One of the principal tasks of reliability analysis is evaluation of importance of individual system components for the system proper work. This investigation is known as importance analysis, and it can be qualitative or quantitative. The former focuses on identification of situations in which a change of component activity causes that system performance will change. Its results can be used to detect situations in which the component is critical for system failure/degradation or to propose procedures that are necessary for system repair. The main objective of quantitative analysis is to estimate and evaluate influence of a change of component activity on system performance, and its outcomes can be used to plan system maintenance or to optimize system availability. This paper focuses on importance analysis of multi-state systems. It introduces new types of logic derivatives and uses them to identify different types of situations in which a system component is critical for system activity. Furthermore, these derivatives are also used to propose a complex framework for quantitative analysis of multi-state systems.**

Keywords—reliability; multi-state system; critical path vectors; critical cut vectors; importance measures; logical differential calculus

I. INTRODUCTION

One of the main tasks of reliability engineering is finding system elements (components) that have the greatest influence on system activity. This task can be done if a mathematical model of the analyzed system is known. Reliability analysis admits several basic models from which the most popular are Binary-State Systems (BSSs) and Multi-State Systems (MSSs) [1–4]. A BSS uses only two states in defining system/ components behavior – state 1 (system/component is working) and state 0 (system/component is failed). One of the main problems of this model is that it requires drawing the line between situations in which the system is considered to be functioning and when it is failed. Therefore, it is mainly used for the analysis of systems in which any deviation from perfect functioning can be viewed as a failure of the system (e.g. nuclear power plants [5], aviation technique [6]). The problem of defining the boundary between system functioning and system failure can be solved using MSSs, which allow defining more than two states in system/components performance. These models are useful in the analysis of systems that can operate under various conditions and, therefore, they are very often used in the analysis of distribution networks [7] or complex socio-technical systems consisting of four basic types of components – hardware, software, organizational and human [8] (typical examples of such systems are healthcare systems studied in papers [9, 10]).

When a mathematical model of a system is created, then its reliability can be investigated. This goal involves several tasks from which identification of components that have the greatest influence on system proper work is one of the most important. This task is known as importance analysis.

Importance analysis has originally been developed for BSSs [11–13]. In this case, it can be used to find situations in which a failure/repair of a system component results in (or contribute to) system failure/repair and to estimate the probability that such situation will occur. The quantification of such situations is done using special indices that are known as Importance Measures (IMs). There exist a lot of IMs for BSSs [14], but the most important and most commonly used are Structural Importance (SI), Birnbaum's Importance (BI), Criticality Importance (CI), and Fussell-Vesely's Importance (FVI) (Table I). In what follows, we will mainly consider the SI, BI, and CI.

TABLE I. IMPORTANCE MEASURES FOR BINARY-STATE SYSTEMS

Importance Measures	Meaning
SI	The SI analyzes system topology and, for a given component, it is defined as a relative number of situations in which a given component is critical for system failure (repair).
BI	The BI takes into account not only system topology but also availabilities of system components and, for a given component, it corresponds to the probability that the component is critical for system failure (repair).
CI	The CI of a given component is computed as the probability that the system failure has been caused by the component failure given that the system is failed.
FVI	The FVI of a given component agrees with the probability that the component contributes to system failure.

The bases of the quantitative importance analysis for MSSs have been developed in [15, 16]. In these papers, several versions of the SI and BI have been proposed. These versions deal with identification of component states that have the greatest influence on system activity. Other types of these IMs have been proposed in [17]. These versions of the SI and BI

This work was supported by the grant of 7th RTD Framework Program No 610425 (RASimAs) and VEGA grant 1/0498/14.

978-1-4673-7186-5/15/$31.00 ©2015 IEEE

can be used to evaluate the total influence of components on the system proper work.

Development of methods for computation of the SI and BI has been considered in [9, 10, 18–20]. In these works, some coincidence between a function defining the dependency of system state on states of its components and logic functions have been investigated, and it has been shown that some tools related to analysis of logic functions can also be used in reliability analysis. One of these tools is logical differential calculus that has originally been developed for analysis of dynamic properties of logic functions. In the considered papers, calculation of the SI and BI using this tool has been proposed. However, work [18] deals only with BSSs and papers [9, 10, 19, 20] only with some special versions of the SI and BI for MSSs.

In this work, we develop approach based on logical differential calculus to create a complex framework for importance analysis of MSSs. We propose a new type of logic derivatives, which can be used to perform comprehensive qualitative analysis of MSSs. Furthermore, these derivatives can be used to compute all versions of the SI and BI proposed in [15–17] and also some other types.

This paper has the next structure. Section II concentrates on bases of reliability analysis. It presents the concept of the system structure function, system availability and importance analysis. Section III focuses on logical differential calculus and its use in reliability analysis. In section IV, the new type of logic derivatives is proposed, and its use in qualitative analysis of MSSs is shown. Finally, section V deals with computation of the SI and BI using the results presented in section IV.

II. Reliability Analysis

A. System Model

Reliability analysis requires creation of system model. One of the basic parts of the model is a mapping defining the dependency between system state and states of its components. This map is known as the structure function. For general MSSs, which admit various numbers of states of individual system components, this function has the following form [2, 3]:

$$\phi(\boldsymbol{x}) = \phi(x_1, x_2,\ldots, x_n):$$
$$\{0,1,\ldots, m_1 -1\}\times\ldots\times\{0,1,\ldots, m_n -1\} \rightarrow \{0,1,\ldots, m -1\}, \quad (1)$$

where n corresponds to the number of system components, m represents number of system states (performance levels), m_i agrees with a count of states of the i-th system component, x_i is a variable representing state of the i-th component, and $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$ is a vector of components states (state vector). Please note that state m -1 (m_i -1) agrees with a situation that the system (component i) is perfectly functioning while state 0 means that the system/component is completely failed.

Homogeneous systems are a special type of MSSs. They allow defining only the same number of states for the system and all its components, therefore, their structure function agrees with the next mapping [2–4, 19]:

$$\phi(\boldsymbol{x}) = \phi(x_1, x_2,\ldots, x_n):$$
$$\{0,1,\ldots, m -1\}^n \rightarrow \{0,1,\ldots, m -1\}, \quad (2)$$

where m denotes the number of system/components states. Please note that this function is formally identical with a Multiple-Valued Logic (MVL) function, which allows us to use some tools of MVL logic in reliability analysis of MSSs [9, 10, 19, 20].

Clearly, if $m = 2$ in definition (2), then we get a mapping representing the structure function of a BSS. In this case, the structure function can be viewed as a Boolean function and, therefore, tools related to analysis of Boolean functions can be used in the analysis of BSSs [18].

The structure function defines system topology. However, it carries no information about probabilities of individual states of system components. This indicates that only its knowledge is insufficient for the analysis of system reliability. In fact, it can be used in the analysis of system topological properties. If we want to investigate other reliability characteristics of the system, such as system availability, mean time to failure or mean time to repair, then the probabilities of individual states of the system components have to be known:

$$p_{i,s} = \Pr\{x_i = s\}, \quad s = 0,1,\ldots, m_i -1. \quad (3)$$

For BSSs, $p_{i,0}$ is known as unavailability of the i-th system component because it corresponds to proportion of time during which the component is failed (unavailable), and it is denoted as q_i. In the similar sense, $p_{i,1}$ coincides with the time during which the component is available, therefore, it is recognized also as the component availability. The availability of binary-state component is simply denoted as p_i.

In the case of MSSs, availability and unavailability of a given component are not very common. Instead of them, s-level probabilities are more often used. These probabilities can be defined in the following way:

$$p_i^{\geq s} = \Pr\{x_i \geq s\},\ q_i^{\geq s} = \Pr\{x_i < s\} = 1 - p_i^{\geq s},$$
$$s = 1,2,\ldots, m_i - 1. \quad (4)$$

Clearly, s-level probability $p_i^{\geq s}$ agrees with the probability that the i-th component is in such state that its performance can satisfy a demand corresponding to state s, while s-level probability $q_i^{\geq s}$ denotes the probability that it cannot satisfy this demand. For example, let us consider a power supply unit that can generate 50 MW, 30 MW, 10 MW, and 0 MW of electricity. This implies that the unit has 4 performance levels from which level 50 MW correspond to state 3, level 30 MW to state 2, level 10 MW to state 1 and level 0 MW to state 0. If there is a demand of at least 20 MW of electricity, then the unit is working if it is at least in state 2. This implies that s-level probabilities have to be computed for $s = 2$ to identify probability that the unit is working/failed in this situation:

$$p_i^{\geq 2} = \Pr\{x_i \geq 2\},\ q_i^{\geq 2} = \Pr\{x_i < 2\}. \quad (5)$$

 978-1-4673-7186-5/15/$31.00

s-level probabilities of system components are used in many approaches of reliability analysis of MSSs [16, 17, 21, 22]. However, they can be derived simply from the probabilities (3) using formula (4). Because of that, a model of a general MSS is usually composed of the structure function (1) and the probabilities of individual states of individual system components (3).

In what follows, we will assume that the components of the system are independent and the system structure function is monotone. The independence means that a change of state of any system component has no effect on states of other components, and the monotonicity of the structure function implies that degradation (improvement) of any system component can cause only system degradation (improvement), i.e. there exists no situation in which system degradation (improvement) results from improvement (degradation) of a system component.

B. System Availability

Knowledge of system structure function (1) and the probabilities (3) can be used to calculate system availability or unavailability. As in the case of s-level probability (4), the availability (unavailability) of a MSS is defined with respect to the minimal state in which the system can satisfy a required demand, i.e.:

$$A^{\geq j} = \Pr\{\phi(\boldsymbol{x}) \geq j\},\ U^{\geq j} = \Pr\{\phi(\boldsymbol{x}) < j\} = 1 - A^{\geq j}, \quad j = 1,2,\ldots,m-1. \tag{6}$$

System availability and unavailability are one of the basic characteristics of MSSs. They can be used to compute other reliability indices, such as mean time to failure or mean time to repair. In the case of BSSs, the availability and unavailability can be defined only for $j = 1$, therefore, they are denoted as A and U instead of $A^{\geq 1}$ and $U^{\geq 1}$ respectively.

Another basic characteristic of BSSs and MSSs is the probability that the system is in state j. Clearly, if we know availabilities $A^{\geq j}$ (unavailabilities $U^{\geq j}$) for all possible j, then the probability that the system is in state j can be calculated in the following way:

$$\Pr\{\phi(\boldsymbol{x}) = j\} = \begin{cases} 1 - A^{\geq 1} & \text{if } j = 0 \\ A^{\geq j} - A^{\geq (j+1)} & \text{if } j \in \{1,2,\ldots,m-2\}, \\ A^{\geq m-1} & \text{if } j = m-1 \end{cases} \quad j = 0,1,\ldots,m-1. \tag{7}$$

In the similar sense, system availability (unavailability) with respect to state j is computed from the probabilities of individual system states as follows:

$$A^{\geq j} = \sum_{h=j}^{m-1} \Pr\{\phi(\boldsymbol{x}) = h\},\ U^{\geq j} = \sum_{h=0}^{j-1} \Pr\{\phi(\boldsymbol{x}) = h\}, \quad j = 1,2,\ldots,m-1. \tag{8}$$

Please note that we have provided such space for system state probability and system availability because some terms used in reliability analysis of MSSs can be defined in two ways depending on whether we focus on system state or on system availability. This will be most evident in importance analysis in which we can investigate influence of system components on a given system state or on a system availability computed with respect to a given system state.

C. Importance Analysis – Qualitative Approach

Availability is an important characteristic of systems, but it does not allow identifying coincidence between system degradation (improvement) and degradation (improvement) of its components. For this purpose other techniques have to be used. One of them is importance analysis.

Importance analysis can be qualitative or quantitative. Both have originally been developed for BSSs. In this case, the basic terms of the qualitative analysis are critical path and cut vectors [14]. A critical path vector for component i is a state vector of the form of $(1_i, \boldsymbol{x}) = (x_1, x_2,\ldots, x_{i-1}, 1, x_{i+1},\ldots, x_n)$ at which a failure of the i-th component causes system failure. Similarly, a critical cut vector for component i is a state vector of the form of $(0_i, \boldsymbol{x}) = (x_1, x_2,\ldots, x_{i-1}, 0, x_{i+1},\ldots, x_n)$ at which a repair of the considered component results in system repair. It is clear that $(1_i, \boldsymbol{x})$ is a critical path vector for component i if and only if the state vector $(0_i, \boldsymbol{x})$ is a critical cut vector for this component. Furthermore, we say that component i is critical at state vector $(\cdot_i, \boldsymbol{x}) = (x_1, x_2,\ldots, x_{i-1}, x_{i+1},\ldots, x_n)$ if $\phi(1_i, \boldsymbol{x}) > \phi(0_i, \boldsymbol{x})$ [14].

Critical path/cut vectors are very important in the qualitative analysis of BSSs because they correspond to situations in which the component makes difference between system functioning and its failure. Some generalizations of this concept for MSSs can be found in [2, 3, 15, 17]. Based on [2, 3, 15], a critical path vector for level j of system availability (i.e. system availability computed with respect to system state j) and for state s of component i can be defined as a state vector $(s_i, \boldsymbol{x}) = (x_1, x_2,\ldots, x_{i-1}, s, x_{i+1},\ldots, x_n)$ such that $\phi(s_i, \boldsymbol{x}) \geq j$ and $\phi((s-1)_i, \boldsymbol{x}) < j$. Similarly, a critical cut vector for level j of system availability and for state s of component i is a state vector $(s_i, \boldsymbol{x})$ such that $\phi(s_i, \boldsymbol{x}) < j$ and $\phi((s+1)_i, \boldsymbol{x}) \geq j$. Based on this approach, state s of component i is critical for degradation (reaching) level j of system availability at state vector $(\cdot_i, \boldsymbol{x})$ if a minor degradation (improvement) of the component state causes that the system degrades below (reaches at least) state j. Furthermore, it is clear that a state vector $(s_i, \boldsymbol{x})$ is a critical path vector for level j of system availability if and only if the state vector $((s-1)_i, \boldsymbol{x})$ is a critical cut vector for the considered system availability level.

Another approach has been considered in [17]. In this paper, a critical path vector has been defined with respect to system state instead of system availability level. So, state vector $(s_i, \boldsymbol{x})$ is a critical path vector for state j of the investigated system and for state s of component i if $\phi(s_i, \boldsymbol{x}) = j$ and $\phi((s-1)_i, \boldsymbol{x}) < j$. The concept of critical cut vectors were not introduced in this paper, but if we want to preserve a coherence with the BSS terminology, then one could expect that a critical cut vector for system state j and for state s of component i should be defined as a state vector $(s_i, \boldsymbol{x})$ such that $\phi(s_i, \boldsymbol{x}) < j$

and $\phi((s+1)_i, \boldsymbol{x}) = j$. Then, it is clear that state s of component i is critical for degradation of (reaching) system state j at state vector $(._i, \boldsymbol{x})$ if a minor degradation (improvement) of the component state causes that the system degrades below (reaches) state j.

D. Importance Analysis – Quantitative Approach

The qualitative analysis concentrates on identification of situations in which a change of component state causes change of system state. The quantitative part of importance analysis estimates the probability that such situations will occur. For this purpose, special indices known as IMs are used. Some of the most commonly used are the SI, BI and CI [1, 14].

The IMs have originally been developed for the analysis of BSSs. In this case, the SI, BI, and CI of a component correlate in some sense with the probability that the component is critical for system failure (repair) (Table I). This indicates that generalizations of these measures for MSSs are based on the assumption that they should coincide with the probability that the considered component or its state is critical for system state or system availability level. Using this assumption, several versions of the SI and BI for homogeneous MSSs have been proposed in [15–17]. These IMs allow:

a) identifying the coincidence between a degradation of a given component state and a deterioration of a given system state/availability level [15, 17],

b) finding component states that have the greatest influence on the whole system (not only on a specified system state/availability level) [16],

c) evaluating the total importance of a given component (not only the importance of a specific component state) on the whole system [17].

Computation of the SI and BI measures in [16, 17] is based on the assumption that the components degrade gradually by one state, i.e. if a component is in state s, then it can degrade only into state s -1. This idea is not unrealistic because even if the component deteriorates from state m_i -1 to state 0, we can assume that it stays in every state from set $\{1,2,\dots, m_i\text{-}1\}$ for very short time. Please note that this assumption will also be taken into account in the rest of this work, which will focus on creating a complex framework for importance analysis of MSSs of any type based on logical differential calculus.

III. Logical Differential Calculus

Logical differential calculus has been developed for analysis of dynamic properties of Boolean and MVL functions. The central term of this tool is logic derivative. There exist several types of logic derivatives from which the most interesting are Direct Partial Logic Derivatives (DPLDs).

A. Direct Partial Logic Derivatives

DPLDs allow identifying situations in which a change of logic variable (Boolean or MVL) results in a change of the investigated logic function (Boolean or MVL). For logic function $f_m(\boldsymbol{x})$, where m denotes number of logic values ($m = 2$ in case of a Boolean function), this derivative is defined with respect to variable x_i as follows [21]:

$$\frac{\partial f_m(j \to h)}{\partial x_i(s \to r)} = \begin{cases} 1 & \text{if } f_m(s_i, \boldsymbol{x}) = j \text{ and } f_m(r_i, \boldsymbol{x}) = h \\ 0 & \text{other} \end{cases}, \quad \text{for } s, r, j, h \in \{0,1,\dots,m\text{-}1\}, s \neq r, j \neq h, \tag{9}$$

where $f_m(a_i, \boldsymbol{x}) = f_m(x_1, x_2,\dots, x_{i-1}, a, x_{i+1},\dots, x_n)$ for $a \in \{s, r\}$. It is clear that nonzero elements of this derivative agree with cases in which change of logic variable x_i from value s to r causes change of logic function value from j to h. Please note that DPLD is a function with Boolean-valued output, i.e. when $m = 2$, then DPLDs are Boolean functions and, in case of the analysis of MVL functions ($m > 2$), DPLDs are functions of MVL variables and with Boolean-valued output.

Definition (9) implies that the next relation exists between DPLDs:

$$\frac{\partial f_m(j \to h)}{\partial x_i(s \to r)} = \frac{\partial f_m(h \to j)}{\partial x_i(r \to s)}, \tag{10}$$

but the principal difference between them is that derivative $\partial f_m(j \to h)/\partial x_i(s \to r)$ can be calculated only at points $(s_i, \boldsymbol{x})$ of the considered function, while the latter is defined only at points $(r_i, \boldsymbol{x})$ (Fig. 1).

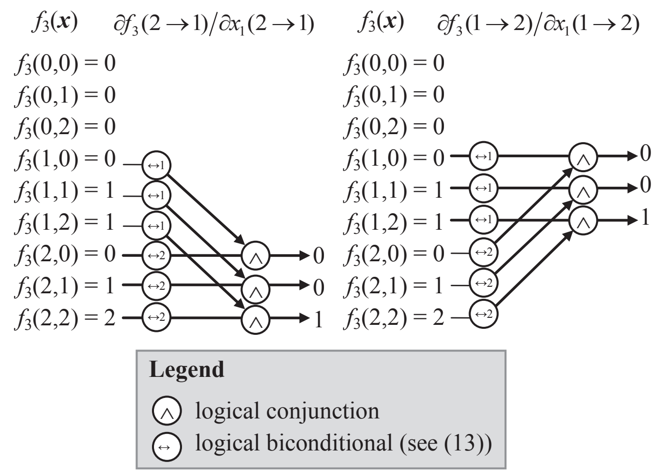

Figure 1. Example of computation and existence of DPLDs for MVL function $f_3(\boldsymbol{x}) = \min(x_1,x_2)$

B. Qualitative Analysis of Homogeneous Systems based on Direct Partial Logic Derivatives

As we mentioned earlier, the structure function of a BSS can be viewed as a Boolean function and the structure of a homogeneous MSS as a MVL function. It follows that DPLDs (9) can be used in the analysis of a BSS and homogenous MSS to find situations in which the studied change of component state results in the considered change of system state. This idea has been presented in [18] for BSSs and in [19, 20] for a special class of homogeneous MSSs.

The monotonicity of the structure function implies that only DPLDs in which $s > r$ and $j > h$ or in which $s < r$ and $j < h$ can be nonzero. The former recognized state vectors at which degradation of the i-th system component from state s to r results in decrease in system state from value j to h, while the latter can be used to find the coincidence between the component improvement and improvement of system state. For example, in the case of BSSs, DPLD $\partial\phi(1\to 0)/\partial x_i(1\to 0)$ identifies situations in which a failure of component i causes system failure, while derivative $\partial\phi(0\to 1)/\partial x_i(0\to 1)$ can be used to find the coincidence between the component repair and system repair. These situations correlate with the meaning of critical path/cut vectors for BSSs and, therefore, the nonzero elements of these two DPLDs correspond to critical path/cut vectors for the i-th system component. So, the critical path vectors for component i agree with the nonzero elements of the next logic expression:

$$\{x_i \leftrightarrow 1\}\frac{\partial\phi(1\to 0)}{\partial x_i(1\to 0)}, \tag{11}$$

and the critical cut vectors with the nonzero elements of the following expression:

$$\{x_i \leftrightarrow 0\}\frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}, \tag{12}$$

where symbol $\leftrightarrow$ denotes logical biconditional, which is defined as follows:

$$\{x_i \leftrightarrow s\} = \begin{cases} 1 & \text{if } x_i = s \\ 0 & \text{other} \end{cases}. \tag{13}$$

Please note that the considered DPLDs have to be combined with expression (13) to find critical path/cut vectors of a BSS because a DPLD with respect to variable x_i does not depend on this variable, while critical path (cut) vectors for component i assume that the component is in state 1 (0), i.e. $x_i = 1$ for critical path vectors and $x_i = 0$ for critical cut vectors. However, if only situations in which the component is critical have to be found, then this can be done simply using the DPLDs without the need to combine them with expression (13) (Table II).

TABLE II. QUALITATIVE ANALYSIS OF BINARY-STATE SYSTEMS BASED ON DIRECT PARTIAL LOGIC DERIVATIVES

Concept	Identification based on DPLDs	
state vectors at which component i is critical	$\frac{\partial\phi(1\to 0)}{\partial x_i(1\to 0)}$	$\frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}$
critical path vectors for component i	$\{x_i \leftrightarrow 1\}\frac{\partial\phi(1\to 0)}{\partial x_i(1\to 0)}$	
critical cut vectors for component i	$\{x_i \leftrightarrow 0\}\frac{\partial\phi(0\to 1)}{\partial x_i(0\to 1)}$	

Use of DPLDs in the qualitative analysis of homogeneous MSSs has been considered in [19, 20]. In these works, only homogeneous MSSs in which a minor degradation of a component can cause only a minor degradation of system state have been considered. This implies that only derivatives $\partial\phi(j\to j-1)/\partial x_i(s\to s-1)$ and $\partial\phi(j\to j+1)/\partial x_i(s\to s+1)$ from the DPLDs analyzing results of a minor degradation and minor improvement of component i can be nonzero. Based on this fact, the critical path vectors for system state j and for state s of component i correspond to the nonzero elements of the next expression:

$$\{x_i \leftrightarrow s\}\frac{\partial\phi(j\to j-1)}{\partial x_i(s\to s-1)}, \quad \text{for } s, j \in \{1,2,\ldots,m\text{-}1\}, \tag{14}$$

while the critical cut vectors for system state j and for state s of component i can be identified as the nonzero elements of the following expression:

$$\{x_i \leftrightarrow s\}\frac{\partial\phi(j-1\to j)}{\partial x_i(s\to s+1)}, \quad \text{for } s \in \{0,1,\ldots,m\text{-}2\} \text{ and } j \in \{1,2,\ldots,m\text{-}1\}. \tag{15}$$

For the considered class of MSSs, it can be simply shown that the critical path (cut) vectors for system state j and for state s of component i agree with the critical path (cut) vectors for level j of system availability and for state s of component i. So, formulae (14) and (15) can also be used to find this type of critical path/cut vectors (Table III).

TABLE III. QUALITATIVE ANALYSIS OF HOMOGENEOUS MULTI-STATE SYSTEMS ADMITTING ONLY MINOR DEGRADATION OF SYSTEM STATE BASED ON DIRECT PARTIAL LOGIC DERIVATIVES

Concept	Identification based on DPLDs
state vectors at which state s of component i is critical for degradation of system state j / level j of system availability	$\frac{\partial\phi(j\to j-1)}{\partial x_i(s\to s-1)}$
state vectors at which state s of component i is critical for reaching system state j / level j of system availability	$\frac{\partial\phi(j-1\to j)}{\partial x_i(s\to s+1)}$
critical path vectors for system state j and for state s of component i / critical path vectors for level j of system availability and for state s of component i	$\{x_i \leftrightarrow s\}\frac{\partial\phi(j\to j-1)}{\partial x_i(s\to s-1)}$
critical cut vectors for system state j and for state s of component i / critical cut vectors for level j of system availability and for state s of component i	$\{x_i \leftrightarrow s\}\frac{\partial\phi(j-1\to j)}{\partial x_i(s\to s+1)}$

C. Quantitative Analysis of Homogeneous Systems based on Direct Partial Logic Derivatives

Since DPLDs can be used to find critical path/cut vectors for BSSs and for the special class of homogeneous MSSs, they can also be used to compute IMs that focus on quantification of situations in which the component is critical for system work. For BSSs, this idea has been considered in [18] where the next formulae have been proposed:

$$\begin{aligned} \mathrm{SI}_i &= \mathrm{TD}\big(\partial\phi(1\to 0)/\partial x_i(1\to 0)\big), \\ \mathrm{BI}_i &= \Pr\{\partial\phi(1\to 0)/\partial x_i(1\to 0)=1\}, \\ \mathrm{CI}_i &= \Pr\{\partial\phi(1\to 0)/\partial x_i(1\to 0)=1\}\frac{q_i}{U} = \mathrm{BI}_i\frac{q_i}{U}, \end{aligned} \tag{16}$$

where TD(.) denotes truth density of the argument interpreted as a function with Boolean-valued output, i.e. it corresponds to the relative number of cases in which the argument is nonzero.

Computation of these measures for a given component state and for a given system state for the considered class of MSSs using DPLDs has been considered in [19, 20]. In these works, the following formulae have been proposed:

$$\begin{aligned} \mathrm{SI}_{i,s}^j &= \mathrm{TD}\big(\partial\phi(j\to j-1)/\partial x_i(s\to s-1)\big), \\ \mathrm{BI}_{i,s}^j &= \Pr\{\partial\phi(j\to j-1)/\partial x_i(s\to s-1)=1\}, \\ \mathrm{CI}_{i,s}^j &= \mathrm{BI}_{i,s}^j\frac{q_{i,s-1}}{U^{\geq j}}. \end{aligned} \tag{17}$$

Based on the coincidence between criticality of state s of component i and DPLDs (Table III), the $\mathrm{SI}_{i,s}^j$ corresponds to the relative number of state vectors at which state s of component i is critical for degradation of state/availability level j of the system, the $\mathrm{BI}_{i,s}^j$ agrees with the probability that state s of component i is critical for degradation of state/availability level j of the system, and the $\mathrm{CI}_{i,s}^j$ identifies the probability that a minor degradation of state s of component i has caused degradation of state/availability level j of the system given that the system is degraded (below state j).

IV. Qualitative Analysis of Multi-State Systems based on Logical Differential Calculus

DPLDs can also be applied to the analysis of non-homogenous MSSs. With respect to the structure function of a non-homogeneous system, this can be achieved based on the following modification of the definition of DPLD:

$$\frac{\partial\phi(j\to h)}{\partial x_i(s\to r)} = \begin{cases} 1 & \text{if } \phi(s_i,\boldsymbol{x}) = j \text{ and } \phi(r_i,\boldsymbol{x}) = h \\ 0 & \text{other} \end{cases}, \tag{18}$$
$$\text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r,\ \ j,h\in\{0,1,\ldots,m\text{-}1\}, j\neq h.$$

Compared to (9), the only difference is in the domain of variables s and r. However, DPLD (18) is still a function with Boolean-valued output, which allows us to compute its truth density or the probability that the DPLD is nonzero.

A. Integrated Direct Partial Logic Derivatives

Let us consider a non-homogeneous MSS in which a minor degradation of a system component can result in deterioration of system state by more than one state. This implies that every DPLD from logic derivatives of the form of $\partial\phi(j\to h)/\partial x_i(s\to s-1)$ where $j>h$ can be nonzero and, therefore, all these derivatives have to be used to find situations in which a minor degradation of the i-th system component causes deterioration of system state j. This fact implies that we have to compute j -1 different DPLDs to identify all situations in which state s of component i is critical for degradation of system state j, i.e. for $h = 0,1,\ldots, j$ -1. The similar is also true for finding state vectors at which state s of component i is critical for degradation of level j of system availability but, in this case, (j -1)(m -1 -j) DPLDs have to be computed, i.e. all DPLDs $\partial\phi(l\to h)/\partial x_i(s\to s-1)$ where $l\geq j$ and $h<j$. However, this can be quite time-consuming. Because of that, we propose a new type of DPLDs that will be named as Integrated Direct Partial Logic Derivatives (IDPLDs). This name reflects that these derivatives are composed of several different DPLDs. Three types of IDPLDs will be distinguished depending on which DPLDs will be combined together.

IDPLDs of type I will be used to find situations in which a change of the i-th component state results in degradation of a given system state:

$$\frac{\partial\phi(j\downarrow)}{\partial x_i(s\to r)} = \bigcup_{h=0}^{j-1}\frac{\partial\phi(j\to h)}{\partial x_i(s\to r)} = \begin{cases} 1 & \text{if } \phi(s_i,\boldsymbol{x}) = j \text{ and } \phi(r_i,\boldsymbol{x}) < j \\ 0 & \text{other} \end{cases}, \tag{19}$$
$$\text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r,\ \ j\in\{1,2,\ldots,m\text{-}1\},$$

or in falling the system into a given state:

$$\frac{\partial\phi(\downarrow j)}{\partial x_i(s\to r)} = \bigcup_{h=j+1}^{m-1}\frac{\partial\phi(h\to j)}{\partial x_i(s\to r)} = \begin{cases} 1 & \text{if } \phi(s_i,\boldsymbol{x}) > j \text{ and } \phi(r_i,\boldsymbol{x}) = j \\ 0 & \text{other} \end{cases}, \tag{20}$$
$$\text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r,\ \ j\in\{0,1,\ldots,m\text{-}2\},$$

where symbol $\bigcup$ denotes logical disjunction/union of DPLDs. If we consider IDPLDs in which $r = s$ -1, then derivative (19) can be used to find state vectors $(\cdot_i, \boldsymbol{x})$ at which state s of the i-th system component is critical for degradation of a given system state and derivative (20) to recognize situations in which state s of component i is critical for falling the system into state j.

IDPLDs of type II will be defined with respect to system structure function in the following way:

978-1-4673-7186-5/15/$31.00 ©2015 IEEE

$$\frac{\partial\phi(\downarrow)}{\partial x_i(s\to r)} = \bigcup_{j=1}^{m-1}\bigcup_{h=0}^{j-1}\frac{\partial\phi(j\to h)}{\partial x_i(s\to r)} = \bigcup_{j=1}^{m-1}\frac{\partial\phi(j\downarrow)}{\partial x_i(s\to r)} = \bigcup_{j=0}^{m-2}\frac{\partial\phi(\downarrow j)}{\partial x_i(s\to r)} = \begin{cases}1 & \text{if } \phi(s_i,\boldsymbol{x})>\phi(r_i,\boldsymbol{x})\\ 0 & \text{other}\end{cases}, \quad \text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r, \tag{21}$$

and they will identify situations in which the studied change of component state results in degradation of the system. If we assume that $r = s$ -1, then this derivative can be used to find state vectors $(._i, \boldsymbol{x})$ at which state s of component i is critical for system degradation (not only for a given system state).

Finally, IDPLDs of type III have been firstly proposed in [24], and they can be defined as follows:

$$\frac{\partial\phi(h_{\geq j}\to h_{<j})}{\partial x_i(s\to r)} = \bigcup_{h_u=j}^{m-1}\bigcup_{h_d=0}^{j-1}\frac{\partial\phi(h_u\to h_d)}{\partial x_i(s\to r)} = \begin{cases}1 & \text{if } \phi(s_i,\boldsymbol{x})\geq j \text{ and } \phi(r_i,\boldsymbol{x})<j\\ 0 & \text{other}\end{cases}, \quad \text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r,\ j\in\{1,2,\ldots,m\text{-}1\}. \tag{22}$$

For $r = s$ -1, these derivatives are useful in finding state vectors $(._i, \boldsymbol{x})$ at which state s of component i is critical for degradation of level j of system availability.

Please note that the similar types of IDPLDs can also be defined for investigation of system improvement, and it can be shown simply that the following relations hold between these IDPLDs and IDPLDs (19) – (22):

$$\frac{\partial\phi(\uparrow j)}{\partial x_i(s\to r)} = \bigcup_{h=0}^{j-1}\frac{\partial\phi(h\to j)}{\partial x_i(s\to r)} = \frac{\partial\phi(j\downarrow)}{\partial x_i(r\to s)}, \quad \text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r,\ j\in\{1,2,\ldots,m\text{-}1\}, \tag{23}$$

$$\frac{\partial\phi(j\uparrow)}{\partial x_i(s\to r)} = \bigcup_{h=j+1}^{m-2}\frac{\partial\phi(j\to h)}{\partial x_i(s\to r)} = \frac{\partial\phi(\downarrow j)}{\partial x_i(r\to s)}, \quad \text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r,\ j\in\{0,1,\ldots,m\text{-}2\}, \tag{24}$$

$$\frac{\partial\phi(\uparrow)}{\partial x_i(s\to r)} = \bigcup_{j=0}^{m-2}\bigcup_{h=j+1}^{m-1}\frac{\partial\phi(j\to h)}{\partial x_i(s\to r)} = \bigcup_{j=1}^{m-1}\frac{\partial\phi(\uparrow j)}{\partial x_i(s\to r)} = \bigcup_{j=0}^{m-2}\frac{\partial\phi(j\uparrow)}{\partial x_i(s\to r)} = \frac{\partial\phi(\downarrow)}{\partial x_i(r\to s)}, \quad \text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r, \tag{25}$$

$$\frac{\partial\phi(h_{<j}\to h_{\geq j})}{\partial x_i(s\to r)} = \bigcup_{h_d=0}^{j-1}\bigcup_{h_u=j}^{m-1}\frac{\partial\phi(h_d\to h_u)}{\partial x_i(s\to r)} = \frac{\partial\phi(h_{\geq j}\to h_{<j})}{\partial x_i(r\to s)}, \quad \text{for } s,r\in\{0,1,\ldots,m_i\text{-}1\}, s\neq r,\ j\in\{1,2,\ldots,m\text{-}1\}. \tag{26}$$

As in the case of DPLDs, IDPLDs investigating system improvement have the same nonzero elements as the corresponding ones studying system degradation, and the only difference is their existence, e.g. the nonzero elements of $\partial\phi(\uparrow j)/\partial x_i(s\to r)$ are same those of $\partial\phi(j\downarrow)/\partial x_i(r\to s)$, but the former DPLD can be computed only at points $(s_i, \boldsymbol{x})$ of the function $\phi(\boldsymbol{x})$ while the latter only at points $(r_i, \boldsymbol{x})$ (Fig. 2).

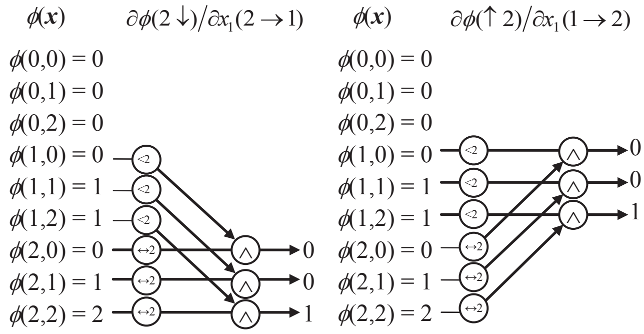

Figure 2. Example of computation and existence of IDPLDs for function $\phi(\boldsymbol{x}) = \min(x_1,x_2)$ for $x_1,x_2 \in \{0,1,2\}$

B. Critical State Vectors

As we mentioned in the previous section, IDPLDs can be used in identification of state vectors at which a given state of component i is critical for system degradation/improvement. Compared to DPLDs, they can be applied simply to any type of MSSs. Furthermore, IDPLDs allow us to introduce some special types of critical state vectors that have no equivalent in the terminology of critical path/cut vectors. These definitions are presented in Table IV for system degradation and in Table V for system improvement. New types of critical state vectors that have not been considered in Table III are denoted by (*).

As we can see in Table IV, IDPLDs of type I and II introduced two new types of critical state vectors. The first one is a critical state vector for falling the system into state j and for state s of component i (or a state vector at which state s of component i is critical for falling the system into state j). This type is useful when we want to find situations in which a minor degradation of a given component state causes that the system falls into a boundary state. (For example, assume that a system can operate if it is at least in state 2 from 4 possible states. Then state 2 is a boundary state because decrease below it causes that the system will be unavailable. This implies that identification of situations in which the system falls into state 2 is very important because it can be used to plan the preventive maintenance of the system, which permits avoiding situations in which the system is unavailable.) The second type of new critical state vectors is a critical state vector for system degradation and for state s of component i (or a state vector at which state s of component i is critical for system degradation), which can be used to find situations in which a minor degradation of state s of component i has influence on system state. Please note that the equivalent types of these two types of critical state vectors can also be defined in terms of system improvement (Table V).

TABLE IV. CRITICAL STATE VECTORS FOR SYSTEM DEGRADATION

Concept	Identification based on IDPLDs
state vectors at which state s of component i is critical for degradation of system state j	$\frac{\partial\phi(j\downarrow)}{\partial x_i(s\to s-1)}$
state vectors at which state s of component i is critical for falling the system into state j (*)	$\frac{\partial\phi(\downarrow j)}{\partial x_i(s\to s-1)}$
state vectors at which state s of component i is critical for system degradation (*)	$\frac{\partial\phi(\downarrow)}{\partial x_i(s\to s-1)}$
state vectors at which state s of component i is critical for degradation of level j of system availability	$\frac{\partial\phi(h_{\geq j}\to h_{<j})}{\partial x_i(s\to s-1)}$
critical path vectors for system state j and for state s of component i	$\{x_i\leftrightarrow s\}\frac{\partial\phi(j\downarrow)}{\partial x_i(s\to s-1)}$
critical state vectors for falling the system into state j and for state s of component i (*)	$\{x_i\leftrightarrow s\}\frac{\partial\phi(\downarrow j)}{\partial x_i(s\to s-1)}$
critical state vectors for system degradation and for state s of component i (*)	$\{x_i\leftrightarrow s\}\frac{\partial\phi(\downarrow)}{\partial x_i(s\to s-1)}$
critical path vectors for level j of system availability and for state s of component i	$\{x_i\leftrightarrow s\}\frac{\partial\phi(h_{\geq j}\to h_{<j})}{\partial x_i(s\to s-1)}$

TABLE V. CRITICAL STATE VECTORS FOR SYSTEM IMPROVEMENT

Concept	Identification based on IDPLDs
state vectors at which state s of component i is critical for reaching system state j	$\frac{\partial\phi(\uparrow j)}{\partial x_i(s\to s+1)}$
state vectors at which state s of component i is critical for improvement of system state j (*)	$\frac{\partial\phi(j\uparrow)}{\partial x_i(s\to s+1)}$
state vectors at which state s of component i is critical for system improvement (*)	$\frac{\partial\phi(\uparrow)}{\partial x_i(s\to s+1)}$
state vectors at which state s of component i is critical for reaching level j of system availability	$\frac{\partial\phi(h_{<j}\to h_{\geq j})}{\partial x_i(s\to s+1)}$
critical cut vectors for system state j and for state s of component i	$\{x_i\leftrightarrow s\}\frac{\partial\phi(\uparrow j)}{\partial x_i(s\to s+1)}$
critical state vectors for improvement of system state j and for state s of component i (*)	$\{x_i\leftrightarrow s\}\frac{\partial\phi(j\uparrow)}{\partial x_i(s\to s+1)}$
critical state vectors for system improvement and for state s of component i (*)	$\{x_i\leftrightarrow s\}\frac{\partial\phi(\uparrow)}{\partial x_i(s\to s+1)}$
critical cut vectors for level j of system availability and for state s of component i	$\{x_i\leftrightarrow s\}\frac{\partial\phi(h_{<j}\to h_{\geq j})}{\partial x_i(s\to s+1)}$

V. QUANTITATIVE ANALYSIS OF MULTI-STATE SYSTEMS BASED ON LOGICAL DIFFERENTIAL CALCULUS

The SI and BI measures are based on quantification of situations in which the considered component is critical for system activity. Based on Table IV and Table V, several types of these measures can be proposed for analysis of coincidence between degradation (improvement) of a given component state and degradation (improvement) of the system. In what follows, we will focus on quantifying the coincidence between component and system degradation.

A. Structural Importance

The SI (17) can be generalized for MSSs of any type using IDPLDs in the following ways:

$$\begin{aligned}\mathrm{SI}_{i,s}^{j\downarrow} &= \mathrm{TD}\big(\partial\phi(j\downarrow)/\partial x_i(s\to s-1)\big),\\ \mathrm{SI}_{i,s}^{\downarrow j} &= \mathrm{TD}\big(\partial\phi(\downarrow j)/\partial x_i(s\to s-1)\big),\\ \mathrm{SI}_{i,s}^{\geq j} &= \mathrm{TD}\big(\partial\phi(h_{\geq j}\to h_{<j})/\partial x_i(s\to s-1)\big).\end{aligned} \tag{27}$$

The first one agrees with the relative number of situations in which a minor degradation of state s of component i causes degradation of system state j. The second corresponds to the relative count of state vectors at which a minor degradation of a given component state results in falling the system into state j. Finally, the third correlates with the proportion of state vectors $(\cdot_i, \boldsymbol{x})$ at which state s of component i is critical for degradation of level j of system availability. Please note that in the case of systems in which a minor degradation of state s of component i can cause only a minor degradation of system state, the next relation holds between SI measures (27) and (17):

$$\mathrm{SI}_{i,s}^{j\downarrow} = \mathrm{SI}_{i,s}^{\downarrow(j-1)} = \mathrm{SI}_{i,s}^{\geq j} = \mathrm{SI}_{i,s}^{j}, \tag{28}$$

which implies that the $\mathrm{SI}_{i,s}^{j}$ in (17) can be used to analyze all dependencies between a component state degradation and system degradation in such systems. However, this is not true for MSSs of other types, which are considered in this part of the paper.

The SI measures (27) are computed using IDPLDs of type I and III. However, IDPLD of type II also identifies some kind of critical state vectors. Therefore, this derivative can be used to introduce the following version of the SI:

$$\mathrm{SI}_{i,s}^{\downarrow} = \mathrm{TD}\big(\partial\phi(\downarrow)/\partial x_i(s\to s-1)\big), \tag{29}$$

which agrees with the relative number of situations in which a minor degradation of state s of component i results in system degradation. As we can see, this measure does not investigate the dependency between a given component state and a given system state, but it focuses on all system states. Moreover, based on definition (21) of IDPLD of type II, one can easy show that this IM can also be computed as follows:

978-1-4673-7186-5/15/$31.00 ©2015 IEEE

$$\mathrm{SI}_{i,s}^{\downarrow} = \sum_{j=1}^{m-1} \mathrm{SI}_{i,s}^{j\downarrow} = \sum_{j=0}^{m-2} \mathrm{SI}_{i,s}^{\downarrow j}. \quad (30)$$

This implies that SI measures (27) can also be used in finding component states with the greatest influence on degradation of the system.

Next, consider SI that allows identifying components that have the greatest influence on a given system state (availability level). IDPLD $\partial\phi(j\downarrow)/\partial x_i(s \to s-1)$ identifies all situations in which state s of component i is critical for degradation of system state j. If we compute this derivative for all possible values of s, i.e. for $s = 1,2,\ldots, m_i$ -1, then we can find all situations in which a minor degradation of the i-th system component causes degradation of system state j. Since there are $(m_i - 1)\prod_{l=1, l\neq i}^{n} m_l$ state vectors at which component i can degrade, i.e. all state vectors of the form of $(s_i, \boldsymbol{x})$ where $s > 0$, this measure can be computed as follows:

$$\mathrm{SI}_i^{j\downarrow} = \frac{\mathrm{TD}\left(\bigcup_{s=1}^{m_i-1} \partial\phi(j\downarrow)/\partial x_i(s \to s-1)\right)}{m_i - 1}, \quad (31)$$

and it corresponds to the relative number of state vectors at which a degradation of component i results in degradation of systems state j. Using some relations between DPLDs and IDPLDs, one can easy show that this formula can be rewritten in the following form:

$$\mathrm{SI}_i^{j\downarrow} = \frac{1}{m_i - 1} \sum_{s=1}^{m_i-1} \mathrm{SI}_{i,s}^{j\downarrow}, \quad (32)$$

which implies that $\mathrm{SI}_i^{j\downarrow}$ can be computed simply if $\mathrm{SI}_{i,s}^{j\downarrow}$ are known for all possible values of s, i.e. for $s \in \{1,2,\ldots, m_i -1\}$.

Similarly, the SI measures for computation of the relative number of situations in which component i is critical for falling the system into state j or for degradation of system availability level j can be computed in the following manner:

$$\begin{aligned} \mathrm{SI}_i^{\downarrow j} &= \frac{\mathrm{TD}\left(\bigcup_{s=1}^{m_i-1} \partial\phi(\downarrow j)/\partial x_i(s \to s-1)\right)}{m_i - 1} \\ &= \frac{1}{m_i - 1} \sum_{s=1}^{m_i-1} \mathrm{SI}_{i,s}^{\downarrow j}, \\ \mathrm{SI}_i^{\geq j} &= \frac{\mathrm{TD}\left(\bigcup_{s=1}^{m_i-1} \partial\phi(h_{\geq j} \to h_{<j}/\partial x_i(s \to s-1)\right)}{m_i - 1} \\ &= \frac{1}{m_i - 1} \sum_{s=1}^{m_i-1} \mathrm{SI}_{i,s}^{\geq j}. \end{aligned} \quad (33)$$

Finally, we can define the total structural importance of a given component as a relative number of situations in which a minor degradation of this component results in system deterioration:

$$\mathrm{SI}_i^{\downarrow} = \frac{1}{m_i - 1} \sum_{s=1}^{m_i-1} \mathrm{SI}_{i,s}^{\downarrow}. \quad (34)$$

Please note that relations between individual types of the SI measures can also be presented in the form of Table VI – Table VIII, which can be very useful in practical investigation of the topological importance of individual system components.

TABLE VI. STRUCTURAL IMPORTANCE MEASURES BASED ON CRITICAL PATH VECTORS FOR GIVEN SYSTEM STATE AND FOR GIVEN COMPONENT STATE

		Component state				**Average**
		1	2	…	m_i -1	
System state	1	$\mathrm{SI}_{i,1}^{1\downarrow}$	$\mathrm{SI}_{i,2}^{1\downarrow}$	…	$\mathrm{SI}_{i,m_i-1}^{1\downarrow}$	$\mathrm{SI}_i^{1\downarrow}$
	2	$\mathrm{SI}_{i,1}^{2\downarrow}$	$\mathrm{SI}_{i,2}^{2\downarrow}$	…	$\mathrm{SI}_{i,m_i-1}^{2\downarrow}$	$\mathrm{SI}_i^{2\downarrow}$
	⋮	⋮	⋮	⋱	⋮	⋮
	m -1	$\mathrm{SI}_{i,1}^{(m-1)\downarrow}$	$\mathrm{SI}_{i,2}^{(m-1)\downarrow}$	…	$\mathrm{SI}_{i,m_i-1}^{(m-1)\downarrow}$	$\mathrm{SI}_i^{(m-1)\downarrow}$
Sum		$\mathrm{SI}_{i,1}^{\downarrow}$	$\mathrm{SI}_{i,2}^{\downarrow}$	…	$\mathrm{SI}_{i,m_i-1}^{\downarrow}$	$\mathrm{SI}_i^{\downarrow}$

TABLE VII. STRUCTURAL IMPORTANCE MEASURES BASED ON CRITICAL STATE VECTORS FOR FALLING THE SYSTEM INTO GIVEN STATE AND FOR GIVEN COMPONENT STATE

		Component state				**Average**
		1	2	…	m_i -1	
System state	0	$\mathrm{SI}_{i,1}^{\downarrow 0}$	$\mathrm{SI}_{i,2}^{\downarrow 0}$	…	$\mathrm{SI}_{i,m_i-1}^{\downarrow 0}$	$\mathrm{SI}_i^{\downarrow 0}$
	1	$\mathrm{SI}_{i,1}^{\downarrow 1}$	$\mathrm{SI}_{i,2}^{\downarrow 1}$	…	$\mathrm{SI}_{i,m_i-1}^{\downarrow 1}$	$\mathrm{SI}_i^{\downarrow 1}$
	⋮	⋮	⋮	⋱	⋮	⋮
	m -2	$\mathrm{SI}_{i,1}^{\downarrow(m-2)}$	$\mathrm{SI}_{i,2}^{\downarrow(m-2)}$	…	$\mathrm{SI}_{i,m_i-1}^{\downarrow(m-2)}$	$\mathrm{SI}_i^{\downarrow(m-2)}$
Sum		$\mathrm{SI}_{i,1}^{\downarrow}$	$\mathrm{SI}_{i,2}^{\downarrow}$	…	$\mathrm{SI}_{i,m_i-1}^{\downarrow}$	$\mathrm{SI}_i^{\downarrow}$

B. Birnbaum's Importance

The only difference between the SI and BI is that the BI takes into account also the probabilities of individual system states. So, using the same approach as in the case of the SI, we can build a complex framework for computation of several types of BI measures that allow:

a) identifying the probability that a given component state is critical for degradation of system state j / for falling the system into state j / for degradation of level j of system availability (the first part of Table IX),

b) computing the probability that a given component state is critical for system degradation (the second part of Table IX),

c) calculating the probability that a given component is critical for degradation of system state j / for falling the system into state j / for degradation of level j of system availability (the third part of Table IX),

d) identifying the probability that a given component is critical for system degradation (the last row in Table IX).

TABLE VIII. STRUCTURAL IMPORTANCE MEASURES BASED ON CRITICAL PATH VECTORS FOR GIVEN SYSTEM AVAILABILITY LEVEL AND FOR GIVEN COMPONENT STATE

		Component state				Average
		1	2	…	m_i -1	
System availability level	1	$\mathrm{SI}_{i,1}^{\geq 1}$	$\mathrm{SI}_{i,2}^{\geq 1}$	…	$\mathrm{SI}_{i,m_i-1}^{\geq 1}$	$\mathrm{SI}_{i}^{\geq 1}$
	2	$\mathrm{SI}_{i,1}^{\geq 2}$	$\mathrm{SI}_{i,2}^{\geq 2}$	…	$\mathrm{SI}_{i,m_i-1}^{\geq 2}$	$\mathrm{SI}_{i}^{\geq 2}$
	⋮	⋮	⋮	⋱	⋮	⋮
	m -1	$\mathrm{SI}_{i,1}^{\geq (m-1)}$	$\mathrm{SI}_{i,2}^{\geq (m-1)}$	…	$\mathrm{SI}_{i,m_i-1}^{\geq (m-1)}$	$\mathrm{SI}_{i}^{\geq (m-1)}$

TABLE IX. BIRNBAUM'S IMPORTANCE MEASURES ANALYZING SYSTEM DEGRADATION

BI	Computation	Meaning
$\mathrm{BI}_{i,s}^{j\downarrow}$	$\Pr\left\{\frac{\partial\phi(j\downarrow)}{\partial x_i(s\to s-1)}=1\right\}$	the probability that state s of component i is critical for degradation of system state j
$\mathrm{BI}_{i,s}^{\downarrow j}$	$\Pr\left\{\frac{\partial\phi(\downarrow j)}{\partial x_i(s\to s-1)}=1\right\}$	the probability that state s of component i is critical for falling the system into state j
$\mathrm{BI}_{i,s}^{\geq j}$	$\Pr\left\{\frac{\partial\phi(h_{\geq j}\to h_{<j})}{\partial x_i(s\to s-1)}=1\right\}$	the probability that state s of component i is critical for degradation of system availability level j
$\mathrm{BI}_{i,s}^{\downarrow}$	$\Pr\left\{\frac{\partial\phi(\downarrow)}{\partial x_i(s\to s-1)}=1\right\} = \sum_{j=1}^{m-1}\mathrm{BI}_{i,s}^{j\downarrow} = \sum_{j=0}^{m-2}\mathrm{BI}_{i,s}^{\downarrow j}$	the probability that state s of component i is critical for system degradation
$\mathrm{BI}_{i}^{j\downarrow}$	$\frac{1}{m_i-1}\sum_{s=1}^{m_i-1}\mathrm{BI}_{i,s}^{j\downarrow}$	the probability that component i is critical for degradation of system state j
$\mathrm{BI}_{i}^{\downarrow j}$	$\frac{1}{m_i-1}\sum_{s=1}^{m_i-1}\mathrm{BI}_{i,s}^{\downarrow j}$	the probability that component i is critical for falling the system into state j
$\mathrm{BI}_{i}^{\geq j}$	$\frac{1}{m_i-1}\sum_{s=1}^{m_i-1}\mathrm{BI}_{i,s}^{\geq j}$	the probability that component i is critical for degradation of system availability level j
$\mathrm{BI}_{i}^{\downarrow}$	$\frac{1}{m_i-1}\sum_{s=1}^{m_i-1}\mathrm{BI}_{i,s}^{\downarrow}$	the probability that component i is critical for system degradation

Please note that the same tables as in the case of the SI measures, i.e. Table VI – Table VIII, can also be applied to present relations between different types of the BI measures. Furthermore, based on definitions (17), the BI measures can be used to compute several types of the CI.

C. *Hand Calculation Example*

Let us consider the simple service system from [25] to illustrate our framework for computation of the IMs for MSSs of any type. In this example, we will focus on the SI and BI measures based on critical path vectors for a given system state and for a given component state, i.e. Table VI. According to [25], the investigated system (Fig. 3) consists of three components – service point 1 (component 1), service point 2 (component 2) and infrastructure (component 3). The system has four possible states: 0 – non-operational (no customer is satisfied), 1 – partially operational (some customers are satisfied), 2 – partially non-operational (some customers are not satisfied), 3 – fully operational (all customers are satisfied). The service points can be only functioning (state 1) or dysfunctional (state 0). The infrastructure is modelled by 4 quality levels, i.e. from 0 (the quality of the infrastructure is poor) to 3 (the quality is perfect). The structure function of this system is defined by Table X. Furthermore, assume that the probabilities of states of system components have values presented in Table XI.

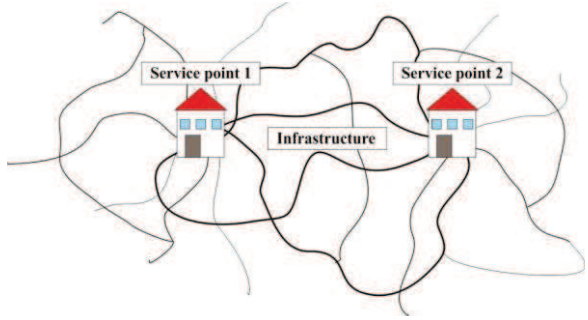

Figure 3. The simple service system and its structure function

TABLE X. STRUCTURE FUNCTION OF THE SERVICE SYSTEM

Components states		x_3			
x_1	x_2	0	1	2	3
0	0	0	0	0	0
0	1	0	1	1	2
1	0	0	1	1	2
1	1	0	2	3	3

TABLE XI. STATE PROBABILITIES OF COMPONENTS OF THE SERVICE SYSTEM

Component	Components state			
	0	1	2	3
1	0.30	0.70	–	–
2	0.20	0.80	–	–
3	0.20	0.60	0.10	0.10

 978-1-4673-7186-5/15/$31.00 ©2015 IEEE

Firstly, analyze the topological importance of individual components. This can be done using the SI measures. For example, if we want to identify influence of a minor degradation of state 1 of component 3 on system state 1, then we can compute IDPLD $\partial\phi(1\downarrow)/\partial x_3(1\to 0)$ and calculate its truth density. The IDPLD has 4 elements, i.e. state vectors (0,0,.), (0,1,.), (1,0,.), and (1,1,.), from which two are nonzero, i.e. (0,1,.) and (1,0,.). This implies that degradation of the 3-rd component from state 1 to 0 causes degradation of state 1 of the system if exactly one of components 1 and 2 is working. This also implies that the truth density of the IDPLD is 0.5, which means that the degradation of state 1 of component 3 results in degradation of system state 1 in a half of all possible situations in which the component can degrade in the considered sense. Similarly, we can calculate all other IDPLDs of the form of $\partial\phi(j\downarrow)/\partial x_i(s\to s-1)$ and then compute their truth densities. This allows us to identify the coincidence between degradation of a given component state and degradation of a given system state. The nonzero elements of the considered IDPLDs are computed in Table XII, and their truth densities, which correspond to the $\mathrm{SI}_{i,s}^{j\downarrow}$, are presented in the central parts of Table XIII and Table XIV.

TABLE XII. NONZERO ELEMENTS OF IDPLDS $\partial\phi(j\downarrow)/\partial x_3(s\to s-1)$

System state	Components and their states				
	1	2	3		
	1	1	1	2	3
1	(.,0,1) (.,0,2)	(0,.,1) (0,.,2)	(0,1,.) (1,0,.)		
2	(.,0,3) (.,1,1)	(0,.,3) (1,.,1)	(1,1,.)		(0,1,.) (1,0,.)
3	(.,1,2) (.,1,3)	(1,.,2) (1,.,3)		(1,1,.)	

According to the data shown in the central parts of Table XIII and Table XIV, degradations of components 1 and 2 have the same influence on every system state, while the 3-rd component degradation has different influences depending on the component state and the system state. According to these data, state 1 of component 3 is the most important for system state 1 while state 2 is the most important for system state 3. One of the interesting results is that state 3 of the considered component has no influence on system state 3.

Next, compute the sum of the $\mathrm{SI}_{i,s}^{j\downarrow}$ shown in the central parts of the considered tables through all possible values of j (the last row of the tables). These values agree with the values of the $\mathrm{SI}_{i,s}^{\downarrow}$ and, therefore, they identify relative counts of situations in which a given component state is critical for the system regardless of system state. According to these data, the most important state of every component is state 1.

In the similar way, calculate the $\mathrm{SI}_{i}^{j\downarrow}$ to find system states for which individual system components are most important. These calculations agree with computation of the mean value of the $\mathrm{SI}_{i,s}^{j\downarrow}$ measures through all possible values of s. These data are in the right column of Table XIII and Table XIV and, as we can see, the first two components have the same influence on every system state while the 3-rd one has the greatest influence on system state 2.

Finally, we can compute the total structural importance of a given component, i.e. the $\mathrm{SI}_{i}^{\downarrow}$, as the relative number of situations in which a degradation of the component results in a degradation of the system. This measure corresponds to the sum of the $\mathrm{SI}_{i}^{j\downarrow}$ through all possible component states, or it can be computed as the average of the $\mathrm{SI}_{i,s}^{\downarrow}$ measures. Please note that values of this IM are presented in the lower right corners of Table XIII and Table XIV, and they indicate that degradations of components 1 and 2 result in system degradation in three quarters of cases in which these components can degrade while a degradation of component 3 causes system degradation only in a half of situations in which it can degrade. Therefore, components 1 and 2 are the most important from topological point of view.

TABLE XIII. STRUCTURAL IMPORTANCE MEASURES BASED ON CRITICAL PATH VECTORS FOR GIVEN SYSTEM STATE AND FOR GIVEN COMPONENT STATE (COMPONENTS 1 AND 2)

		Component state	Average
		1	
System state	1	0.25	0.25
	2	0.25	0.25
	3	0.25	0.25
Sum		0.75	**0.75**

TABLE XIV. STRUCTURAL IMPORTANCE MEASURES BASED ON CRITICAL PATH VECTORS FOR GIVEN SYSTEM STATE AND FOR GIVEN COMPONENT STATE (COMPONENT 3)

		Component state			Average
		1	2	3	
System state	1	0.50	0	0	0.1667
	2	0.25	0	0.50	0.2500
	3	0	0.25	0	0.0833
Sum		0.75	0.25	0.50	**0.50**

IDPLDs computed in Table XII can also be used to calculate the $\mathrm{BI}_{i,s}^{j\downarrow}$, $\mathrm{BI}_{i,s}^{\downarrow}$, $\mathrm{BI}_{i}^{j\downarrow}$, and $\mathrm{BI}_{i}^{\downarrow}$. For this task, states probabilities of individual system components (Table XI) are needed. For example, the $\mathrm{BI}_{1,3}^{1\downarrow}$, which identifies the probability that degradation of state 1 of component 3 is critical for system state 1 can be computed as the probability that IDPLD $\partial\phi(1\downarrow)/\partial x_3(1\to 0)$ is nonzero:

$$\begin{aligned}\mathrm{BI}_{1,3}^{1\downarrow} &= \Pr\{\partial\phi(1\downarrow)/\partial x_3(1\to 0)=1\} = p_{1,0}p_{2,1}+p_{1,1}p_{2,0} \\ &= 0.3*0.8+0.7*0.2=0.38,\end{aligned} \tag{35}$$

and, therefore, a minor degradation of the considered state of component 3 results in degradation of system state 1 with the probability of 0.38. In the similar way, other $\mathrm{BI}_{i,s}^{j\downarrow}$ can be computed (Table XV – Table XVII). Next, we can compute the $\mathrm{BI}_{i,s}^{\downarrow}$, which can be used to find component state whose degradation causes degradation of the system with the most probability, as the sum of all $\mathrm{BI}_{i,s}^{j\downarrow}$ through all possible system states. Similarly, the $\mathrm{BI}_{i}^{j\downarrow}$, which analyzes the probability that degradation of component *i* causes degradation of system state *j*, can be computed as the average of $\mathrm{BI}_{i,s}^{j\downarrow}$ measures through all possible component states. Finally, we can use these results to compute the total influence of the component on the system state, i.e. the $\mathrm{BI}_{i}^{\downarrow}$ (the lower right corners of Table XV – Table XVII).

According to data in Table XV – Table XVII, the most important components are the 1-st and 2-nd since their degradations result in system degradation with the probability of 0.80. Next, we can state that the all three components have the greatest influence on system state 2 and that state 1 of every component is the most important since its degradation results in system degradation with the most probability.

VI. CONCLUSION

In this paper, a complex framework for qualitative and quantitative importance analysis of MSSs was considered. This framework is based on new types of logic derivatives introduced in this paper. These derivatives were named as IDPLDs since they are composed of several DPLDs. IDPLDs can be used in identification of various types of critical state vectors, i.e. critical path/cut vectors for system state/ availability level *j* and for state *s* of component *i*, critical state vectors for falling the system into/improvement of state *j* and for state *s* of component *i*, critical state vectors for system degradation/improvement and for state *s* of component *i*. These critical state vectors can be then used to investigate the importance of system components on system activity. Next, in this work, we combined several approaches used in importance analysis. The result of this is a complex approach that can be used to find components with the greatest influence on system degradation/improvement, or components with the greatest influence on degradation/improvement of a specific system state (availability level), or component states with the greatest impact on system degradation/improvement, or component states with the greatest impact on degradation/improvement of a specific system state (availability level).

TABLE XV. BIRNBAUM'S IMPORTANCE MEASURES BASED ON CRITICAL PATH VECTORS FOR GIVEN SYSTEM STATE AND FOR GIVEN COMPONENT STATE (COMPONENT 1)

		Component state	Average
		1	
System state	1	0.14	0.14
	2	0.50	0.50
	3	0.16	0.16
Sum		0.80	**0.80**

TABLE XVI. BIRNBAUM'S IMPORTANCE MEASURES BASED ON CRITICAL PATH VECTORS FOR GIVEN SYSTEM STATE AND FOR GIVEN COMPONENT STATE (COMPONENT 2)

		Component state	Average
		1	
System state	1	0.21	0.21
	2	0.45	0.45
	3	0.14	0.14
Sum		0.80	**0.80**

TABLE XVII. BIRNBAUM'S IMPORTANCE MEASURES BASED ON CRITICAL PATH VECTORS FOR GIVEN SYSTEM STATE AND FOR GIVEN COMPONENT STATE (COMPONENT 3)

		Component state			Average
		1	2	3	
System state	1	0.38	0	0	0.1267
	2	0.56	0	0.38	0.3133
	3	0	0.56	0	0.1867
Sum		0.94	0.56	0.38	**0.6267**

REFERENCES

[1] M. Rausand and A. Høyland, System Reliability Theory, 2nd ed. Haboken, NJ: John Wiley & Sons, Inc., 2004.

[2] A. Lisnianski and G. Levitin, Multi-state System Reliability. Assessment, Optimization and Applications. Singapore, SG: World Scientific, 2003.

[3] A. Lisnianski, I. Frenkel, and Y. Ding, Multi-state System Reliability Analysis and Optimization for Engineers and Industrial Managers. London, UK: Springer-Verlag London Ltd., 2010.

[4] B. Natvig, Multistate Systems Reliability Theory with Applications. Chichester, UK: John Wiley & Sons, Ltd, 2011.

[5] Y. Watanabe, T. Oikawa, and K. Muramatsu, "Development of the DQFM method to consider the effect of correlation of component failures in seismic PSA of nuclear power plant," Reliability Engineering & System Safety, vol. 79, no. 3, pp. 265–279, Mar. 2003.

[6] B. Nystrom, L. Austrin, N. Ankarback, and E. Nilsson, "Fault tree analysis of an aircraft electric power supply system to electrical actuators," in Probabilistic Methods Applied to Power Systems, 2006. PMAPS 2006. International Conference on, 2006, pp. 1–7.

978-1-4673-7186-5/15/$31.00 ©2015 IEEE

[7] W.-C. Yeh, “A simple approach to search for all d-MCs of a limited-flow network,” Reliability Engineering & System Safety, vol. 71, no. 1, pp. 15–19, Jan. 2001.

[8] E. Zio, “Reliability engineering: Old problems and new challenges,” Reliability Engineering & System Safety, vol. 94, no. 2, pp. 125–141, Feb. 2009.

[9] E. Zaitseva, “Reliability analysis methods for healthcare system,” in Human System Interactions (HSI), 2010 3rd Conference on, 2010, pp. 211–216.

[10] E. Zaitseva, V. Levashenko, and M. Rusin, “Reliability analysis of healthcare system,” in 2011 Federated Conference on Computer Science and Information Systems, FedCSIS 2011, 2011, pp. 169–175.

[11] Z. W. Birnbaum, “On the importance of different components in a multicomponent system,” in Multivariate Analysis, vol. 2, P. R. Krishnaiah, Ed. New York, NY: Academic Press, 1969, pp. 581–592.

[12] W. E. Vesely, “A time-dependent methodology for fault tree evaluation,” Nuclear Engineering and Design, vol. 13, no. 2, pp. 337–360, Aug. 1970.

[13] J. B. Fussell, “How to hand-calculate system reliability and safety characteristics,” IEEE Transactions on Reliability, vol. R-24, no. 3, pp. 169–174, Aug. 1975.

[14] W. Kuo and X. Zhu, Importance Measures in Reliability, Risk, and Optimization: Principles and Applications. Chichester, UK: Wiley, 2012.

[15] D. A. Butler, “A complete importance ranking for components of binary coherent systems, with extensions to multi-state systems,” Naval Research Logistics Quarterly, vol. 26, no. 4, pp. 565–578, Dec. 1979.

[16] W. S. Griffith, “Multistate reliability models,” Journal of Applied Probability, vol. 17, no. 3, pp. 735–744, Sep. 1980.

[17] S. Wu, “Joint importance of multistate systems,” Computers & Industrial Engineering, vol. 49, no. 1, pp. 63–75, Aug. 2005.

[18] E. N. Zaitseva and V. G. Levashenko, “Importance analysis by logical differential calculus,” Automation and Remote Control, vol. 74, no. 2, pp. 171–182, Feb. 2013.

[19] E. Zaitseva, “Importance analysis of multi-state system by tools of differential logical calculus,” in Reliability, Risk, and Safety. Theory and Application, vol. 3, C. Guedes Soares, R. Briš, and S. Martorell, Eds. London, UK: CRC Press, 2010, pp. 1579–1584.

[20] E. Zaitseva and V. Levashenko, “Multiple-valued logic mathematical approaches for multi-state system reliability analysis,” Journal of Applied Logic, vol. 11, no. 3, pp. 350–362, Sep. 2013.

[21] G. Levitin, L. Podofillini, and E. Zio, “Generalised importance measures for multi-state elements based on performance level restrictions,” Reliability Engineering & System Safety, vol. 82, no. 3, pp. 287–298, Dec. 2003.

[22] E. Zio, L. Podofillini, and G. Levitin, “Estimation of the importance measures of multi-state elements by Monte Carlo simulation,” Reliability Engineering & System Safety, vol. 86, no. 3, pp. 191–204, Dec. 2004.

[23] S. N. Yanushkevich, D. M. Miller, V. P. Shmerko, and R. S. Stankovic, Decision Diagram Techniques for Micro- and Nanoelectronic Design Handbook, vol. 2. Boca Raton, FL: CRC Press, 2005.

[24] M. Kvassay, E. Zaitseva, V. Levashenko, and J. Kostolny, “Minimal cut vectors and logical differential calculus,” in 2014 IEEE 44th International Symposium on Multiple-Valued Logic, 2014, pp. 167–172.

[25] M. Kvassay, E. Zaitseva, and V. Levashenko, “Minimal cut sets and direct partial logic derivatives in reliability analysis,” in Safety and Reliability: Methodology and Applications - Proceedings of the European Safety and Reliability Conference, ESREL 2014, 2015, pp. 241–248.